

COMBINATORIAL DEGREE BOUND FOR TORIC IDEALS OF HYPERGRAPHS

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ABSTRACT. Associated to any hypergraph is a toric ideal encoding the algebraic relations among its edges. We study these ideals and the combinatorics of their minimal generators, and derive general degree bounds for both uniform and non-uniform hypergraphs in terms of balanced hypergraph bicolourings, separators, and splitting sets. In turn, this provides complexity bounds for algebraic statistical models associated to hypergraphs. As two main applications, we recover a well-known complexity result for Markov bases of arbitrary 3-way tables, and we show that the defining ideal of the tangential variety is generated by quadratics and cubics in cumulant coordinates.

1. INTRODUCTION

The edge subring of a hypergraph H is the monomial subalgebra parametrized by the edges of H . We derive a general degree bound for the minimal generators of its defining ideal, I_H , in terms of the structure of the underlying hypergraph.

Let H be a hypergraph on $V = \{1, \dots, n\}$ with edge set E . Each edge $e_i \in E$ of size d encodes a squarefree monomial $x^{e_i} := \prod_{j \in e_i} x_j$ of degree d in the polynomial ring $k[x_1, \dots, x_n]$. The *edge subring* of the hypergraph H , denoted by $k[H]$, is the following monomial subring:

$$k[H] := k[x^{e_i} : e_i \in E(H)].$$

The *toric ideal* of $k[H]$, denoted I_H , is the kernel of the monomial map $\phi_H : k[t_{e_i}] \rightarrow k[H]$ defined by $\phi_H(t_{e_i}) = x^{e_i}$. The ideal I_H encodes the algebraic relations among the edges of the hypergraph. For the special case where H is a graph, generating sets of the toric ideal of $k[H]$ have been studied combinatorially in [12, 13], [15], [19], and [20, 21].

The motivation for studying toric ideals I_H is threefold. First, explicit results that relate I_H to combinatorial properties of H have existed only for graphs. Second, I_H is related to the Rees algebra $\mathcal{R}(\mathcal{I}(H))$ of the edge ideal of H : in case when H is a graph, the presentation ideal of $\mathcal{R}(\mathcal{I}(H))$ is completely determined by generators of I_H and the syzygies of the edge ideal $\mathcal{I}(H)$. Third, these toric ideals correspond to Markov bases for algebraic statistical models; this connection is outlined at the end of the Introduction. A starting point of our work is the fact that combinatorial signatures of generators of I_H are balanced edge sets of H . Balanced edge sets on uniform hypergraphs were introduced in [14], and are referred to as monomial walks.

This paper is based on the fact that the ideal I_H is generated by binomials $f_{\mathcal{E}}$ arising from primitive balanced edge sets \mathcal{E} of H (See Proposition 3.1, a generalization of [14, Theorem 2.8]). A *balanced edge set* of H is a multiset of bicolored edges $\mathcal{E} = \mathcal{E}_{\text{blue}} \sqcup \mathcal{E}_{\text{red}}$ satisfying the following balancing condition: for each vertex v covered by \mathcal{E} , the number of red edges

containing v equals the number of blue edges containing v , that is,

$$(*) \quad \deg_{blue}(v) = \deg_{red}(v).$$

A binomial $f_{\mathcal{E}}$ arises from \mathcal{E} if it can be written as

$$f_{\mathcal{E}} = \prod_{e \in \mathcal{E}_{blue}} t_e - \prod_{e' \in \mathcal{E}_{red}} t_{e'}.$$

Note that while H is a simple hypergraph (it contains no multiple edges), \mathcal{E} allows repetition of edges. In addition, the balanced edge set \mathcal{E} is *primitive* if there exists no other balanced edge set $\mathcal{E}' = \mathcal{E}'_{blue} \sqcup \mathcal{E}'_{red}$ such that $\mathcal{E}'_{blue} \subsetneq \mathcal{E}_{blue}$ and $\mathcal{E}'_{red} \subsetneq \mathcal{E}_{red}$; this is the usual definition of an element in the Graver basis of I_H . If H is a uniform hypergraph, a balanced edge set is called a *monomial walk* to conform with the terminology in [20, 21] and [14].

In what follows, we give two general degree bounds for generators of I_H (Section 5), study the combinatorics of splitting sets and reducibility (defined in Section 3), and explore implications to algebraic statistics throughout. Section 4 focuses on indispensable binomials, i.e. binomials that are members of every minimal generating set of I_H . Proposition 4.1 gives a combinatorial sufficient condition for determining whether a binomial $f \in I_H$ is indispensable. Consequently, the Graver basis is the unique minimal generating set of I_H for any 2-regular hypergraph (Proposition 4.2). In particular, this means that the Graver basis is equal to the universal Gröbner basis, although the defining matrix need not be unimodular.

Theorem 5.1 is a combinatorial criterion for the ideal of a uniform hypergraph to be generated in degree at most $d \geq 2$. The criterion is based on decomposable balanced edge sets, separators, and splitting sets; see Definitions 3.2 and 3.3. Our result generalizes the well-known criterion for the toric ideal of a graph to be generated in degree 2 from [12] and [20, 21]. Splitting sets translate and extend the constructions used in [12] and [20, 21] to hypergraphs and arbitrary degrees. Theorem 5.3 provides a more general result for non-uniform hypergraphs.

In algebraic statistics, any log-linear statistical model corresponds to a toric variety whose defining ideal gives a Markov basis for the model (cf. Fundamental Theorem of Markov bases [6], [8]). Since these varieties, by definition, have a monomial parametrization, we can also associate to any log-linear model \mathcal{M} with a square-free parameterization a (non-uniform) hypergraph $H_{\mathcal{M}}$. By Proposition 3.1, Markov moves for the model \mathcal{M} are described by balanced edge sets of $H_{\mathcal{M}}$: if \mathcal{E} is a balanced edge set of $H_{\mathcal{M}}$, then a Markov move on a fiber of the model corresponds to replacing the set of red edges in \mathcal{E} by the set of blue edges in \mathcal{E} . Our degree bounds give a bound for the *Markov complexity* (Markov width) of the model \mathcal{M} . For general references on Markov complexity of classes for some log-linear models, the reader should refer to [4], [5], [8, Chapter 1, §2] and [10].

We apply our combinatorial criteria to recover a well-known result in algebraic statistics from [4] in Corollary 4.8. Finally, we study the Markov complexity of a set of models from [18] called hidden subset models. Namely, Theorem 5.6 says that the ideal associated to the image of $\text{Tan}((\mathbb{P}^1)^n)$ in higher cumulants is generated by quadratics and cubics.

2. PRELIMINARIES AND NOTATION

We remind the reader that all hypergraphs in this paper are simple, that is, they contain no multiple edges. In contrast, balanced edge sets of hypergraphs are not, since the binomials

arising from the sets need not be squarefree. Therefore, for the purpose of this manuscript, we will refer to a balanced edge set as a *multiset* of edges, with implied vertex set; and, as usual, $V(\mathcal{E})$ denotes the vertex set contained in the edges in \mathcal{E} .

For the remainder of this short section, we will clear the technical details and notation we need for the proofs that follow.

A multiset, M , is an ordered pair (A, f) such that A is a set and f is a function from A to $\mathbb{N}_{>0}$ that records the multiplicity of each of the elements of A . For example, the multiset $M = (\{1, 2\}, f)$ with $f(1) = 1$ and $f(2) = 3$ represents $M = \{1, 2, 2, 2\}$ where ordering doesn't matter. We will commonly use the latter notation.

Given a multiset $M = (A, f)$, the *support* of M is $\text{supp}(M) := A$, and its size is $|M| := \sum_{a \in A} f(a)$. For two multisets $M_1 = (A, f_1)$ and $M_2 = (B, f_2)$, we say $M_2 \subseteq M_1$ if $B \subseteq A$ and for all $b \in B$, $f_2(b) \leq f_1(b)$. M_2 is a proper submultiset of M_1 if $B \subsetneq A$, or there exists a $b \in B$ such that $f_2(b) < f_1(b)$.

Unions, intersections, and relative complements of multisets are defined in the canonical way:

$$M_1 \cup M_2 := (A \cup B, g) \text{ where } g(a) = \begin{cases} f_1(a) & \text{if } a \in A \setminus B, \\ f_2(a) & \text{if } a \in B \setminus A, \\ \max(f_1(a), f_2(a)) & \text{if } a \in A \cap B; \end{cases}$$

$$M_1 \cap M_2 := (A \cap B, g) \text{ where } g(a) = \min(f_1(a), f_2(a));$$

$$M_1 - M_2 := (C, g), \text{ where } g(a) = \begin{cases} f_1(a) & \text{if } a \in A \setminus B, \\ f_1(a) - f_2(a) & \text{otherwise.} \end{cases}$$

and $C = A \setminus B \cup \{a \in A \cap B \mid f_1(a) - f_2(a) > 0\}$

Note that the support of the union (intersection) of two multisets is the union (intersection) of their supports. Finally, we define a *sum* of M_1 and M_2 :

$$M_1 \sqcup M_2 := (A \cup B, g) \text{ where } g(a) = \begin{cases} f_1(a) & \text{if } a \in A \setminus B, \\ f_2(a) & \text{if } a \in B \setminus A, \\ f_1(a) + f_2(a) & \text{if } a \in A \cap B \end{cases}.$$

If $M_1 \sqcup M_2$ is a balanced edge set, then the notation $M_1 \sqcup_b M_2$ will be used to record the bicoloring of $M_1 \sqcup M_2$: edges in M_1 are blue, and edges in M_2 are red.

Finally, the number of edges in a hypergraph H containing a vertex v will be denoted by $\deg(v; H)$. For a bicolored multiset $M := M_{\text{blue}} \sqcup_m M_{\text{red}}$, the blue degree $\deg_{\text{blue}}(v; M)$ of a vertex v is defined to be $\deg(v; M_{\text{blue}})$. The red degree $\deg_{\text{red}}(v; M)$ is defined similarly.

3. SPLITTING SETS AND REDUCIBLE EDGE SETS

The aim of this section is to lay the combinatorial groundwork for studying toric ideals of hypergraphs. In particular, we explicitly state what it combinatorially means for a binomial arising from a monomial walk to be generated by binomials of a smaller degree. We begin by describing the binomial generators of I_H . Unless otherwise stated, H need not be uniform.

Proposition 3.1. *Every binomial in the toric ideal of a hypergraph corresponds to a balanced edge set. In particular, the toric ideal I_H is generated by primitive balanced edge sets.*

Proof. Suppose \mathcal{E} is a balanced multiset of edges over H . Define a binomial $f_{\mathcal{E}} \in k[t_e : e \in E(H)]$ as follows:

$$f_{\mathcal{E}} = \prod_{e \in \mathcal{E}_{\text{blue}}} t_e - \prod_{e' \in \mathcal{E}_{\text{red}}} t_{e'}.$$

The balancing condition (*) ensures that $f_{\mathcal{E}}$ is in the kernel of the map ϕ_H .

The second claim is immediate. \square

Motivated by the application of reducible simplicial complexes to understand the Markov bases of hierarchical log-linear models [7], we now introduce notions of reducibility and separators for balanced edge sets. For simplicity, we will often abuse notation and use H to denote the edge set of H .

Definition 3.2. A balanced edge set \mathcal{E} is said to be *reducible with separator* S , $\text{supp}(S) \subseteq \text{supp}(\mathcal{E})$, and *decomposition* (Γ_1, S, Γ_2) , if there exist balanced edge sets $\Gamma_1 \neq \mathcal{E}$ and $\Gamma_2 \neq \mathcal{E}$ with $S \neq \emptyset$ such that $S = \Gamma_{1,\text{red}} \cap \Gamma_{2,\text{blue}}$, $\mathcal{E} = \Gamma_1 \sqcup \Gamma_2$, and the following coloring conditions hold: $\Gamma_{1,\text{red}}, \Gamma_{2,\text{red}} \subseteq \mathcal{E}_{\text{red}}$ and $\Gamma_{1,\text{blue}}, \Gamma_{2,\text{blue}} \subseteq \mathcal{E}_{\text{blue}}$.

We say that S is *proper* with respect to (Γ_1, S, Γ_2) if S is a proper submultiset of both $\Gamma_{1,\text{red}}$ and $\Gamma_{2,\text{blue}}$.

If S is not proper, then S is said to be *blue* with respect to (Γ_1, S, Γ_2) if $\Gamma_{1,\text{red}} = S$, and *red* with respect to (Γ_1, S, Γ_2) if $\Gamma_{2,\text{blue}} = S$.

Figure 1 shows an example of a reducible balanced edge set \mathcal{E} . The separator is proper and consists of the single green edge e_s ; it appears twice in the balanced edge set \mathcal{E} , once as a blue edge and once as a red edge. Figure 2 shows a reducible balanced edge set where the separator, consisting of the two green edges e_1 and e_2 , is not proper. As before, the separator edges appear twice in the balanced edge set.

If H is a hypergraph and \mathcal{E} is a balanced edge set with $\text{supp}(\mathcal{E}) \subseteq H$, given a multiset S with $\text{supp}(S) \subseteq H$, we can construct a new balanced edge set in the following manner:

$$\mathcal{E} + S := (\mathcal{E}_{\text{blue}} \sqcup S) \sqcup_m (\mathcal{E}_{\text{red}} \sqcup S).$$

Definition 3.3. Let H be a hypergraph. Let \mathcal{E} be a balanced edge set with size $2n$ such that $\text{supp}(\mathcal{E}) \subseteq H$. A non-empty multiset S with $\text{supp}(S) \subseteq H$ is a *splitting set* of \mathcal{E} with decomposition (Γ_1, S, Γ_2) if $\mathcal{E} + S$ is reducible with separator S and decomposition (Γ_1, S, Γ_2) . S is said to be a *blue* (*red*, resp.) *splitting set with respect* to (Γ_1, S, Γ_2) , if S is a blue (*red*, resp.) separator of $\mathcal{E} + S$ with respect to (Γ_1, S, Γ_2) .

S is a *proper splitting set* of \mathcal{E} if there exists a decomposition (Γ_1, S, Γ_2) of $\mathcal{E} + S$ such that S is a proper separator with respect to (Γ_1, S, Γ_2) .

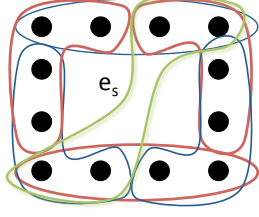


FIGURE 1. Reducible balanced edge set. The green edge e_s is the separator.

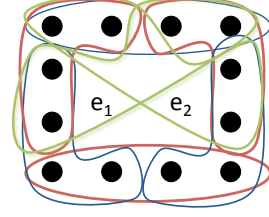


FIGURE 2. Reducible balanced edge set with an improper separator. The separator consists of green edges e_1 and e_2 .

Example 3.4 (Group-based Markov model). Let $V_1 = \{x_1, x_2, x_3, x_4\}$, $V_2 = \{y_1, y_2, y_3, y_4\}$, and $V_3 = \{z_1, z_2, z_3, z_4\}$. Let V be the disjoint union of V_1 , V_2 , and V_3 . Let H be the 3-uniform hypergraph with vertex set V and edge set:

$$\begin{array}{llll} e_{111} = \{x_1, y_1, z_1\} & e_{122} = \{x_1, y_2, z_2\} & e_{133} = \{x_1, y_3, z_3\} & e_{144} = \{x_1, y_4, z_4\} \\ e_{221} = \{x_2, y_2, z_1\} & e_{212} = \{x_2, y_1, z_2\} & e_{243} = \{x_2, y_4, z_3\} & e_{234} = \{x_2, y_3, z_4\} \\ e_{331} = \{x_3, y_3, z_1\} & e_{342} = \{x_3, y_4, z_2\} & e_{313} = \{x_3, y_1, z_3\} & e_{324} = \{x_3, y_2, z_4\} \\ e_{441} = \{x_4, y_4, z_1\} & e_{432} = \{x_4, y_3, z_2\} & e_{423} = \{x_4, y_2, z_3\} & e_{414} = \{x_4, y_1, z_4\} \end{array}$$

The hypergraph H has applications in algebraic phylogenetics: it represents the parametrization of a particular group-based model from [17, Example 25].

Consider the monomial walk

$$\mathcal{W} = \{e_{324}, e_{111}, e_{243}, e_{432}\} \sqcup_m \{e_{122}, e_{313}, e_{234}, e_{441}\}.$$

Let $S = \{e_{133}, e_{212}\}$. Then S is a splitting set of \mathcal{W} with decomposition (Γ_1, S, Γ_2) where

$$\begin{aligned} \Gamma_1 &= \{e_{111}, e_{243}, e_{432}\} \sqcup_m \{e_{133}, e_{212}, e_{441}\} \\ \Gamma_2 &= \{e_{133}, e_{212}, e_{324}\} \sqcup_m \{e_{122}, e_{313}, e_{234}\}. \end{aligned}$$

The decomposition (Γ_1, S, Γ_2) encodes binomials in I_H that generate $f_{\mathcal{W}}$:

$$f_{\mathcal{W}} = t_{e_{324}}(t_{e_{111}} t_{e_{243}} t_{e_{432}} - t_{e_{133}} t_{e_{212}} t_{e_{441}}) + t_{e_{441}}(t_{e_{133}} t_{e_{212}} t_{e_{324}} - t_{e_{122}} t_{e_{313}} t_{e_{234}}).$$

The previous example illustrates the algebraic interpretation of a splitting set. Notice there is a correspondence between monomials in $k[t_{e_i}]$ and multisets of edges of H . We will write $E(t_{e_{i_1}}^{a_1} t_{e_{i_2}}^{a_2} \cdots t_{e_{i_l}}^{a_l})$ for the multiset $(\{e_{i_1}, \dots, e_{i_l}\}, f)$ where

$$\begin{aligned} f : \{e_{i_1}, \dots, e_{i_l}\} &\rightarrow \mathbb{N} \\ e_{i_j} &\mapsto a_j. \end{aligned}$$

Thus the support of $E(t_{e_{i_1}}^{a_1} t_{e_{i_2}}^{a_2} \cdots t_{e_{i_l}}^{a_l})$ corresponds to the support of the monomial $t_{e_{i_1}}^{a_1} t_{e_{i_2}}^{a_2} \cdots t_{e_{i_l}}^{a_l}$.

If $f_{\mathcal{E}} = u - v \in I_H$ is the binomial arising from the balanced edge set \mathcal{E} , then a monomial s corresponds to a splitting set S if and only if there exist two binomials $u_1 - v_1, u_2 - v_2 \in I_H$ such that $us = u_1 u_2$, $vs = v_1 v_2$ and $s = \gcd(v_2, u_1)$. In this case, the decomposition of $\mathcal{E} + S$ is (Γ_1, S, Γ_2) where $\Gamma_1 = E(u_1) \sqcup_m E(v_1)$ and $\Gamma_2 = E(u_2) \sqcup_m E(v_2)$.

For a balanced edge set, \mathcal{E} , the existence of a spitting set determines whether the binomial $f_{\mathcal{E}} \in I_H$ can be written as the linear combination of two binomials $f_{\Gamma_1}, f_{\Gamma_2} \in I_H$. While, in general, the existence of a splitting set does not imply $\deg(f_{\Gamma_1}), \deg(f_{\Gamma_2}) < \deg(f_{\mathcal{E}})$, if H is uniform and the splitting set is proper, then the following lemma holds.

Lemma 3.5. *Let H be a uniform hypergraph and let \mathcal{W} be a monomial walk with $\text{supp}(\mathcal{W}) \subseteq H$ and $|\mathcal{W}| = 2n$. If S is a proper splitting set of \mathcal{W} , then there exists a decomposition (Γ_1, S, Γ_2) of $\mathcal{W} + S$ such that $|\Gamma_1| < |\mathcal{W}|$ and $|\Gamma_2| < |\mathcal{W}|$.*

Proof. Let S be a proper splitting set of \mathcal{W} . By definition, there exists a decomposition (Γ_1, S, Γ_2) of $\mathcal{W} + S$, such that S is a proper submultiset of $\Gamma_{1_{\text{red}}}$ and $\Gamma_{2_{\text{blue}}}$.

Let $|\Gamma_1| = 2n_1$ and $|\Gamma_2| = 2n_2$. Since $\mathcal{W} + S = \Gamma_1 \sqcup \Gamma_2$, it follows that $|\mathcal{W} + S| = |\Gamma_1| + |\Gamma_2|$. Then, $2n + 2|S| = 2n_1 + 2n_2$, which implies $2n - 2n_1 = 2n_2 - 2|S|$. But S being a proper submultiset of $\Gamma_{2_{\text{blue}}}$ gives that $n_2 > |S|$, which, in turn, implies that $n > n_1$. By a similar argument, $n > n_2$. Thus $|\Gamma_1| < |\mathcal{W}|$ and $|\Gamma_2| < |\mathcal{W}|$. \square

4. INDISPENSABLE BINOMIALS

A binomial f in a toric ideal I is *indispensable* if f or $-f$ belongs to every binomial generating set of I . Indispensable binomials of toric ideals were introduced by Takemura et al, and are studied in [1], [2], [3], [13], [15].

Proposition 4.1. *Let H be a hypergraph. Let \mathcal{E} be a balanced edge set with $\text{supp}(\mathcal{E}) \subseteq H$. Let $f_{\mathcal{E}}$ be the binomial arising from \mathcal{E} . If there does not exist a splitting set of \mathcal{E} , then $f_{\mathcal{E}}$ is an indispensable binomial of I_H .*

Proof. Suppose \mathcal{E} is not indispensable. Then there is a binomial generating set of I_H , $\mathcal{G} = \{f_1, \dots, f_n\}$, such that $f_{\mathcal{E}} \notin \mathcal{G}$ and $-f_{\mathcal{E}} \notin \mathcal{G}$.

Since $f_{\mathcal{E}} = f_{\mathcal{E}}^+ - f_{\mathcal{E}}^- \in I_H$, there is a $f_i = f_i^+ - f_i^- \in \mathcal{G}$ such that f_i^+ or f_i^- divides $f_{\mathcal{E}}^+$. Without loss of generality, assume $f_i^+ | f_{\mathcal{E}}^+$. Since f_i is a binomial in I_H , f_i arises from a monomial walk \mathcal{E}_i on H .

Let $S = \mathcal{E}_{i_{\text{red}}}$. Let $\Gamma_1 = \mathcal{E}_i$ and $\Gamma_2 = \Gamma_{2_{\text{blue}}} \sqcup_m \Gamma_{2_{\text{red}}}$ where

$$\begin{aligned} \Gamma_{2_{\text{blue}}} &= ((\mathcal{E}_{\text{blue}} - \mathcal{E}_{i_{\text{blue}}}) \sqcup \mathcal{E}_{i_{\text{red}}}) \\ \Gamma_{2_{\text{red}}} &= \mathcal{E}_{\text{red}}. \end{aligned}$$

Since $f_i^+ | f_{\mathcal{E}}^+$, the multiset $\mathcal{E}_{i_{\text{blue}}} \subseteq \mathcal{E}_{\text{blue}}$, and thus $\Gamma_1 \sqcup \Gamma_2 = \mathcal{E} + S$. By construction, $\Gamma_{1_{\text{red}}} \cap \Gamma_{2_{\text{blue}}} = S$. Therefore S is a splitting set of \mathcal{E} . \square

If every Graver basis element of a binomial ideal I_H is indispensable, then the Graver basis of I_H is the unique minimal generating set of I_H . Propositions 4.2 and 4.6 describe two classes of hypergraphs where this is the case. In particular, for these hypergraphs, the universal Gröbner basis of I_H is a minimal generating set.

Proposition 4.2. *If H is a 2-regular uniform hypergraph, then the Graver basis of I_H is the unique minimal generating set of I_H .*

For the proof of Proposition 4.2, we make use of Proposition 3.2 in [14] which concerns balanced edge sets that are pairs of perfect matchings.

Definition 4.3. A *matching* on a hypergraph $H = (V, E)$ is a subset $M \subseteq E$ such that the elements of M are pairwise disjoint. A matching is called *perfect* if $V(M) = V$.

Proof of Proposition 4.2. Let \mathcal{G} be the Graver basis of I_H and let $f \in \mathcal{G}$. Since every element of \mathcal{G} is binomial, f arises from a primitive monomial walk \mathcal{W} with $\text{supp}(\mathcal{W}) \subseteq H$.

Let $M_b = \text{supp}(\mathcal{W}_{red})$ and $M_r = \text{supp}(\mathcal{W}_{blue})$. By primitivity of \mathcal{W} , the intersection $M_r \cap M_b = \emptyset$. Since \mathcal{W} satisfies condition (*) and H is 2-regular, if $e_1, e_2 \in M_b$ and $e_1 \cap e_2 \neq \emptyset$, then $e_1 \in M_r$ or $e_2 \in M_r$, which would contradict the primitivity of \mathcal{W} . So M_b and M_r are two edge-disjoint perfect matchings on $V(\mathcal{W})$. By Proposition 3.2 in [14], \mathcal{W} contains no multiple edges, i.e. $\mathcal{W} = M_b \sqcup_m M_r$. Furthermore, since H is 2-regular, the edge set of the subhypergraph induced by $V(\mathcal{W})$ is $M_b \cup M_r$.

Suppose S is a splitting set of \mathcal{W} with decomposition (Γ_1, S, Γ_2) . By the correspondence between primitive monomial walks and primitive binomials, there exists a primitive monomial walk Γ such that $\Gamma_{blue} \subseteq \Gamma_{1_{blue}}$ and $\Gamma_{red} \subseteq \Gamma_{1_{red}}$ (if Γ_1 is primitive, then $\Gamma = \Gamma_1$). By Proposition 3.2 in [14], Γ must be a pair of perfect matchings on $V(\Gamma)$. This means Γ is a proper balanced edge set of \mathcal{W} , a contradiction. Therefore, by Proposition 4.1, $f_{\mathcal{W}}$ is indispensable. Since every element in the Graver basis of I_H is indispensable, there is no generating set of I_H strictly contained in the Graver basis, and the claim follows. \square

Definition 4.4. A k -uniform hypergraph $H = (V, E)$ is k -partite if there exists a partition of V into k disjoint subsets, V_1, \dots, V_k , such that each edge in E contains exactly one vertex from each V_i .

Lemma 4.5. Let $H = (V, E)$ be a k -uniform k -partite hypergraph with $E = E_b \sqcup E_r$ and $E_b \cap E_r = \emptyset$. If there exists a V_i , $1 \leq i \leq k$, such that $\deg(v; E_r) = \deg(v; E_b) = 1$ for all $v \in V_i$, then a monomial walk \mathcal{W} with support E is primitive only if \mathcal{W} contains no multiple edges.

Proof. Follows from the proof of necessity of Proposition 3.2 in [14]. \square

Proposition 4.6. Let $H = (V, E)$ be a k -uniform k -partite hypergraph. If there exists a V_i such that $\deg(v; E) = 2$ where for all $v \in V_i$, then the Graver basis of I_H is the unique minimal generating set of I_H .

Proof. The proof is similar to the proof of Proposition 4.2. Note that while H may not be 2-regular, one of its parts, V_i , is ‘locally’ 2-regular, and thus restricts the structure of monomial walks on H . In particular, Lemma 4.5 ensures that M_r and M_b , are edge-disjoint perfect matchings on $V(\mathcal{W})|_{V_i}$, and the rest of the proof follows immediately. \square

Example 4.7 (No 3-way interaction). The toric ideal of the hypergraph H in Figure 3 corresponds to the hierarchical log-linear model for no 3-way interaction on $2 \times 2 \times 2$ contingency tables. This statistical model is a common example in algebraic statistics [8, Example 1.2.7]. Since there is exactly one primitive monomial walk \mathcal{W} on H that travels through 8 edges, $I_H = (f_{\mathcal{W}})$.

For $2 \times 3 \times 3$ contingency tables with no 3-way interaction, the hypergraph corresponding to this log-linear model has 18 edges. The hypergraph in this case is $H = (V, E)$ where $V = \{x_{00}, x_{01}, x_{02}, x_{10}, x_{11}, x_{12}, y_{00}, y_{01}, y_{02}, y_{10}, y_{11}, y_{12}, z_{00}, z_{01}, z_{02}, z_{10}, z_{11}, z_{12}, z_{20}, z_{21}, z_{22}\}$ and the

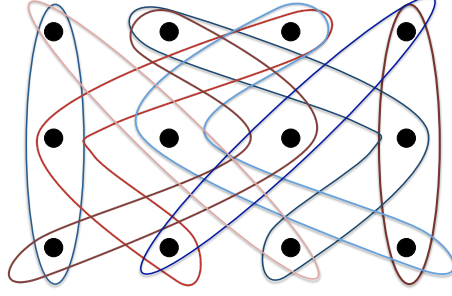


FIGURE 3.

edge set is:

$$\begin{array}{lll}
 e_{000} = \{x_{00}, y_{00}, z_{00}\} & e_{001} = \{x_{00}, y_{01}, z_{01}\} & e_{002} = \{x_{00}, y_{02}, z_{02}\} \\
 e_{010} = \{x_{01}, y_{00}, z_{10}\} & e_{011} = \{x_{01}, y_{01}, z_{11}\} & e_{012} = \{x_{01}, y_{02}, z_{12}\} \\
 e_{020} = \{x_{02}, y_{00}, z_{20}\} & e_{021} = \{x_{02}, y_{01}, z_{21}\} & e_{022} = \{x_{02}, y_{02}, z_{22}\} \\
 e_{100} = \{x_{10}, y_{10}, z_{00}\} & e_{101} = \{x_{10}, y_{11}, z_{01}\} & e_{102} = \{x_{10}, y_{12}, z_{02}\} \\
 e_{110} = \{x_{11}, y_{10}, z_{10}\} & e_{111} = \{x_{11}, y_{11}, z_{11}\} & e_{112} = \{x_{11}, y_{12}, z_{12}\} \\
 e_{120} = \{x_{12}, y_{10}, z_{20}\} & e_{121} = \{x_{12}, y_{11}, z_{21}\} & e_{122} = \{x_{12}, y_{12}, z_{22}\}
 \end{array}$$

Let \mathcal{W} be the primitive monomial walk

$$\mathcal{W} = \{e_{000}, e_{101}, e_{011}, e_{112}, e_{022}, e_{120}\} \sqcup_m \{e_{100}, e_{001}, e_{111}, e_{012}, e_{122}, e_{220}\}.$$

Every remaining edge H that does not appear in \mathcal{W} is not contained in $V(\mathcal{W})$, thus it can be easily verified that there does not exist a splitting set of \mathcal{W} , so by Proposition 4.1, $f_{\mathcal{W}}$ is indispensable. In fact, H satisfies the condition of Proposition 4.6 and thus every binomial in I_H corresponding to a primitive monomial walk is indispensable.

From the above discussion, we can see that if a uniform hypergraph H contains an induced subhypergraph H_s that is 2-regular and there exists a bicoloring such that with this bicoloring H_s is also a balanced edge set, then the maximum degree of any minimal generating set of I_H is at least $|E(H_s)|/2$.

A similar statement holds for k -uniform, k -partite hypergraphs with vertex partition $V = \cup_{i=1}^k V_i$. Namely, if H contains an induced subhypergraph H_s that is 2-regular on V_i (i.e., H satisfies the conditions of Proposition 4.6) and there exists a bicoloring such that with this bicoloring H_s is a balanced edge set (e.g., H_s is a pair of disjoint perfect matchings), then the maximum degree of any minimal generating set of I_H is at least $|E(H_s)|/2$.

Recall that degree bounds on minimal generators give a Markov complexity bound for the corresponding log-linear model in algebraic statistics. This allows us to recover a well-known result:

Corollary 4.8 (Consequence of Theorem 1.2 in [4]; see also Theorem 1.2.17 in [8]). *The Markov complexity for the no 3-way interaction model on $3 \times r \times c$ contingency tables grows arbitrarily large as r and c increase.*

Proof. For the no 3-way interaction model on $2 \times r \times c$ contingency tables, we can construct a primitive binomial f_{H_s} of degree $2 \cdot \min(r, c)$ in its defining toric ideal by taking a cycle

of length $\min(r, c)$ on the bipartite graph $K_{r,c}$. (We remind the reader that this is precisely how $f_{\mathcal{W}}$ is constructed in Example 4.7). By noting that the hypergraph associated to this binomial H_s is an induced subhypergraph of the hypergraph associated to the $3 \times r \times c$ case and that H_s is 2-regular in one of the partitions, the claim follows by Proposition 4.6. \square

5. GENERAL DEGREE BOUNDS AND AN APPLICATION

For uniform hypergraphs, balanced edge sets are referred to as monomial walks. In the previous sections, we saw that splitting sets of \mathcal{W} translate to algebraic operations on the binomials $f_{\mathcal{W}}$, providing a general construction for rewriting a high-degree binomial in terms of binomials corresponding to shorter walks. This, along with Lemma 3.5, is the key to the general degree bound result.

Theorem 5.1. *Given a k -uniform hypergraph H , the toric ideal I_H is generated in degree at most d if and only if for every primitive monomial walk \mathcal{W} of length $2n > 2d$, with $\text{supp}(\mathcal{W}) \subseteq H$, one of the following two conditions hold:*

i) *there exists a proper splitting set S of \mathcal{W} ,*

or

ii) *there is a finite sequence of pairs, $(S_1, R_1), \dots, (S_N, R_N)$, such that*

- S_1 and R_1 are blue and red splitting sets of \mathcal{W} of size less than n with decompositions $(\Gamma_{1_1}, S_1, \Gamma_{2_1})$ and $(\Upsilon_{1_1}, R_1, \Upsilon_{2_1})$,
- S_{i+1} and R_{i+1} are blue and red splitting sets of $\mathcal{W}_i = \Gamma_{2_{i_{\text{blue}}}} \sqcup_m \Upsilon_{1_{i_{\text{red}}}}$ of size less than n with decompositions $(\Gamma_{1_{i+1}}, S_{i+1}, \Gamma_{2_{i+1}})$ and $(\Upsilon_{1_{i+1}}, R_{i+1}, \Upsilon_{2_{i+1}})$, and,
- $S_N \cap R_N \neq \emptyset$ or there exists a proper splitting set of \mathcal{W}_N .

Proof of necessity (\Rightarrow). Let H be a k -uniform hypergraph whose toric ideal I_H is generated in degree at most d . Let \mathcal{W} be a primitive monomial walk of length $2n > 2d$. Let $p_{\mathcal{W}} = u - v$ be the binomial that arises from \mathcal{W} . Since I_H is generated in degree at most d , there exist primitive binomials of degree at most d , $(u_1 - v_1), \dots, (u_s - v_s) \in k[t_{e_i}]$, and $m_1, \dots, m_s \in k[t_{e_i}]$, such that

$$p_{\mathcal{W}} = m_1(u_1 - v_1) + m_2(u_2 - v_2) + \dots + m_s(u_s - v_s).$$

By expanding and reordering so that $m_1 u_1 = u_w$, $m_s v_s = v_w$, and $m_i v_i = m_{i+1} u_{i+1}$ for all $i = 1, \dots, s-1$, we may and will assume that m_1, \dots, m_s are monomials.

If $\gcd(m_i, m_{i+1}) \neq 1$ for some i , we can add the terms $m_i(u_i - v_i)$ and $m_{i+1}(u_{i+1} - v_{i+1})$ to get a new term, $m'_i(u'_i - v'_i)$, where $m'_i = \gcd(m_i, m_{i+1})$ and $(u'_i - v'_i)$ is a binomial of I_H of degree less than n . Continuing recursively in the manner, we have

$$p_{\mathcal{W}} = m'_1(u'_1 - v'_1) + m'_2(u'_2 - v'_2) + \dots + m'_r(u'_r - v'_r)$$

where $m'_1 u'_1 = u'_w$, $m'_r v'_r = v'_w$, $m'_i v'_i = m'_{i+1} u'_{i+1}$, $\gcd(m'_i, m'_{i+1}) = 1$ for all $i = 1, \dots, r-1$, and $\deg(u'_i - v'_i) < n$ for all $i = 1, \dots, r$. For convenience, we will drop the superscripts and write

$$p_w = m_1(u_1 - v_1) + m_2(u_2 - v_2) + \dots + m_r(u_r - v_r).$$

Case 1: $r = 2$. In this case, $p_{\mathcal{W}} = m_1(u_1 - v_1) + m_2(u_2 - v_2)$. Let

$$\begin{aligned}\Gamma_1 &:= E(u_1) \sqcup_m E(v_1) \\ \Gamma_2 &:= E(u_2) \sqcup_m E(v_2) \\ S &:= E(v_1) \cap E(u_2) = E(\gcd(v_1, u_2)).\end{aligned}$$

We want to show (Γ_1, S, Γ_2) is a decomposition of $\mathcal{W} + S$. Since $S = \Gamma_{1_{red}} \cap \Gamma_{2_{blue}}$, $\Gamma_{1_{blue}} \subseteq \mathcal{W}_{blue}$, and $\Gamma_{2_{red}} \subseteq \mathcal{W}_{red}$, we only need to show $\mathcal{W} + S = \Gamma_1 \sqcup \Gamma_2$, $\Gamma_{2_{red}} \subseteq (\mathcal{W} + S)_{red}$, and $\Gamma_{2_{blue}} \subseteq (\mathcal{W} + S)_{blue}$. First, notice the following equalities hold:

$$\begin{aligned}\mathcal{W} + S &= (\mathcal{W}_{blue} \sqcup S) \sqcup (\mathcal{W}_{red} \sqcup S) = E(u) \sqcup S \sqcup E(v) \sqcup S \\ &= E(m_1 u_1) \sqcup S \sqcup E(m_2 v_2) \sqcup S = E(m_1) \sqcup E(u_1) \sqcup S \sqcup E(m_2) \sqcup E(v_2) \sqcup S.\end{aligned}$$

Let $s \in k[t_{e_i}]$ be the monomial such that $E(s) = S$, so $s = \gcd(v_1, u_2)$. The equality $m_1 v_1 = m_2 u_2$ implies $m_1(\frac{v_1}{s}) = m_2(\frac{u_2}{s})$. Now, $\frac{v_1}{s}$ and $\frac{u_2}{s}$ are clearly relatively prime, and by the assumptions on $p_{\mathcal{W}}$, m_1 and m_2 are relatively prime. This means the equality $m_1(\frac{v_1}{s}) = m_2(\frac{u_2}{s})$ implies $m_1 = \frac{u_2}{s}$ and $m_2 = \frac{v_1}{s}$. Thus,

$$\begin{aligned}\Gamma_1 \sqcup \Gamma_2 &= E(u_1) \sqcup E(v_1) \sqcup E(u_2) \sqcup E(v_2) \\ &= E(u_1) \sqcup E(\frac{v_1}{s}) \sqcup S \sqcup E(v_2) \sqcup E(\frac{u_2}{s}) \sqcup S \\ &= E(u_1) \sqcup E(m_2) \sqcup S \sqcup E(v_2) \sqcup E(m_1) \sqcup S.\end{aligned}$$

Consequently, $\mathcal{W} + S = \Gamma_1 \sqcup \Gamma_2$.

Notice the equality $m_2 = \frac{v_1}{s}$ also implies $\Gamma_{1_{red}} = E(v_1) = E(m_2) \sqcup S$. This means $\Gamma_{1_{red}} \subseteq (E(m_2 u_2) \sqcup S) = (\mathcal{W}_{red} \sqcup S) = (\mathcal{W} + S)_{red}$. By a similar observation, $\Gamma_{2_{blue}} \subseteq (\mathcal{W} + S)_{blue}$.

Case 2: $r = 2N + 1$. For $1 < i < N$, let

$$\begin{aligned}\Gamma_{1_i} &= E(u_i) \sqcup_m E(v_i) \\ \Gamma_{2_i} &= E(m_{i+1} u_{i+1}) \sqcup_m E(m_{2N-i+2} v_{2N-i+2}) \\ S_i &= E(v_i) \cap E(m_{i+1} u_{i+1}) = E(\gcd(v_i, m_{i+1} u_{i+1})) = E(v_i).\end{aligned}$$

For $1 < i < N$, let

$$\begin{aligned}\Upsilon_{1_i} &= E(m_i u_i) \sqcup_m E(m_{2N-i+1} v_{2N-i+1}) \\ \Upsilon_{2_i} &= E(u_{2N-i+2}) \sqcup_m E(v_{2N-i+2}) \\ R_i &= E(m_{2N-i+1} v_{2N-i+1}) \cap E(u_{2N-i+2}) \\ &= E(\gcd(m_{2N-i+1} v_{2N-i+1}, u_{2N-i+2})) = E(u_{2N-i+2}).\end{aligned}$$

One can follow the proof of Case 1) to see that S_1 and R_1 are splitting sets of \mathcal{W} , and S_{i+1} and R_{i+1} are splitting sets of $\mathcal{W}_i = E(m_{i+1} u_{i+1}) \sqcup_m E(m_{2N-i+1} v_{2N-i+1})$ for $i = 1, \dots, N-1$. Furthermore, by definition, they are blue and red splitting sets (resp.) of size less than $2n$.

Since $\mathcal{W}_{N-1_{blue}} = \Gamma_{2_{N-1_{blue}}}$ and $\mathcal{W}_{N-1_{red}} = \Upsilon_{1_{N-1_{red}}}$, the binomial arising from the walk on \mathcal{W}_{N-1} is

$$m_N u_N - m_{N+2} v_{N+2} = m_N(u_N - v_N) + m_{N+1}(u_{N+1} - v_{N+1}) + m_{N+2}(u_{N+2} - v_{N+2}).$$

Choose $e \in H$ such that $t_e \mid m_{N+1}$, then $t_e \mid v_N$ and $t_e \mid u_{N+2}$. But since $S_N = E(v_N)$ and $R_N = E(u_{N+2})$, $e \in S_N$ and $e \in R_N$, so $S_N \cap R_N \neq \emptyset$.

Case 3: $r = 2N + 2$. For $1 < i < N$, let

$$\begin{aligned}\Gamma_{1_i} &= E(u_i) \sqcup_m E(v_i) \\ \Gamma_{2_i} &= E(m_{i+1}u_{i+1}) \sqcup_m E(m_{2N-i+3}v_{2N-i+3}) \\ S_i &= E(v_i) \cap E(m_{i+1}u_{i+1}) = E(\gcd(v_i, m_{i+1}u_{i+1})) = E(v_i).\end{aligned}$$

For $1 < i < N$, let

$$\begin{aligned}\Upsilon_{1_i} &= E(m_i u_i) \sqcup_m E(m_{2N-i+2} v_{2N-i+2}) \\ \Upsilon_{2_i} &= E(u_{2N-i+3}) \sqcup_m E(v_{2N-i+3}) \\ R_i &= E(m_{2N-i+2} v_{2N-i+2}) \cap E(u_{2N-i+3}) \\ &= E(\gcd(m_{2N-i+2} v_{2N-i+2}, u_{2N-i+3})) = E(u_{2N-i+3}).\end{aligned}$$

We can follow the proof of Case 1) to see that S_1 and R_1 are splitting sets of \mathcal{W} , and S_{i+1} and R_{i+1} are splitting sets of $\mathcal{W}_i = E(m_{i+1}u_{i+1}) \sqcup_m E(m_{2N-i+2}v_{2N-i+2})$ for $i = 1, \dots, N-1$. Furthermore, by definition, they are blue and red (resp.) splitting sets of size less than n . Since $\mathcal{W}_{N_{blue}} = \Gamma_{2N_{blue}}$ and $\mathcal{W}_{N_{red}} = \Upsilon_{1N_{red}}$, the binomial arising from \mathcal{W}_N is

$$m_{N+1}u_{N+1} - m_{N+2}v_{N+2} = m_{N+1}(u_{N+1} - v_{N+1}) + m_{N+2}(u_{N+2} - v_{N+2})$$

which is exactly case 1), which means there exists a proper splitting set of \mathcal{W}_N . □

Proof of sufficiency (\Leftarrow). Assume every primitive monomial walk \mathcal{W} of length $2n > 2d$ with $\text{supp}(\mathcal{W}) \subset H$ satisfies *i*) or *ii*). Let $p_{\mathcal{W}} = u - v$ be a generator of I_H which arises from the monomial walk \mathcal{W} on H .

To show that $I_H = [I_H]_{\leq d}$, we proceed by induction on the degree of $p_{\mathcal{W}}$. If $\deg p_{\mathcal{W}} = 2$, then $p_{\mathcal{W}} \in [I_H]_{\leq d}$. So assume $\deg p_{\mathcal{W}} = n > d$ and every generator of I_H of degree less than n is in $[I_H]_{\leq d}$. Since the size of \mathcal{W} is greater than $2d$, either condition *i*) holds or condition *ii*) holds.

Suppose *i*) holds. By Lemma 3.5, there exists a decomposition of \mathcal{W} , (Γ_1, S, Γ_2) , such that $|\Gamma_1| < |\mathcal{W}|$ and $|\Gamma_2| < |\mathcal{W}|$. Let $p_{\Gamma_1} = u_1 - v_1$ ($p_{\Gamma_2} = u_2 - v_2$, respectively) be the binomial that arises from Γ_1 (Γ_2 , respectively). Let $m_1 = u/u_1$ and $m_2 = v/v_2$.

What remains to be shown is that $p_{\mathcal{W}} = m_1 p_{\Gamma_1} + m_2 p_{\Gamma_2}$, that is, $u - v = m_1(u_1 - v_1) + m_2(u_2 - v_2)$. However, it is clear that $u = m_1 u_1$ and $v = m_2 v_2$, so it suffices to show is that $m_1 v_1 = m_2 u_2$, or equivalently, $E(m_1 v_1) = E(m_2 u_2)$.

Let $s \in k[t_{e_i}]$ be the monomial such that $E(s) = S$. Then

$$\Gamma_1 \sqcup \Gamma_2 = (E(u_1) \sqcup E(\frac{v_1}{s}) \sqcup S) \sqcup (E(\frac{u_2}{s}) \sqcup S \sqcup E(v_2))$$

and

$$\mathcal{W} + S = (E(m_1) \sqcup E(u_1) \sqcup S) \sqcup (E(m_2) \sqcup E(v_2) \sqcup S).$$

Thus, since $\mathcal{W} + S = \Gamma_1 \sqcup \Gamma_2$,

$$E(m_1) \sqcup E(m_2) = E(\frac{v_1}{s}) \sqcup E(\frac{u_2}{s}),$$

which in turn implies

$$m_1 m_2 = (\frac{v_1}{s})(\frac{u_2}{s}).$$

Since \mathcal{W} is primitive and the coloring conditions on (Γ_1, S, Γ_2) imply $E(\frac{v_1}{s}) \subseteq \mathcal{W}_{red}$ and $E(m_1) \subseteq \mathcal{W}_{blue}$, the monomials m_1 and $\frac{v_1}{s}$ are relatively prime. A similar argument shows m_2 and $\frac{u_2}{s}$ are relatively prime. Thus, $m_1 = \frac{u_2}{s}$ and $m_2 = \frac{v_1}{s}$, and consequently, $E(m_1 v_1) = E(m_2 u_2)$ and $p_w = m_1 p_{\Gamma_1} + m_2 p_{\Gamma_2}$.

Since $\deg p_{\Gamma_1}, \deg p_{\Gamma_2} < n$, the induction hypothesis applied to p_{Γ_1} and p_{Γ_2} shows that $p_w \in [I_H]_{\leq d}$.

Now suppose *ii*) holds. For i from 1 to N , let $p_{\Gamma_{1_i}} = u_i - v_i$ and $p_{\Gamma_{2_i}} = y_i - z_i$ be the binomials arising from Γ_{1_i} and Γ_{2_i} . Let $w_{i_b} - w_{i_r}$ be the binomial arising from the walk \mathcal{W}_i and let $p_w = w_{0_b} - w_{0_r}$. For $1 \leq i \leq N$, let $m_i = w_{(i-1)_b}/u_i$, and $q_i = w_{(i-1)_r}/z_i$. Then

$$p_w = \sum_{i=1}^N m_i(u_i - v_i) + w_{N_b} - w_{N_r} + \sum_{i=1}^N q_{N+1-i}(y_{N+1-i} - z_{N+1-i}).$$

The preceding claim follows from three observations: (1) by construction, $w_{0_b} = m_1 u_1$ and $w_{0_r} = q_1 z_1$; (2) by the definition of \mathcal{W}_N , $w_{N_b} = m_N v_N$ and $w_{N_r} = q_N y_N$; and (3) by the definitions of m_i, q_i , and the walk \mathcal{W}_i , $m_i v_i = m_{i+1} u_{i+1}$ and $q_{i+1} z_{i+1} = q_i y_i$ for $1 \leq i \leq N-1$. As a consequence of the size conditions on the splitting sets of \mathcal{W}_i , the linear combination $\sum_{i=1}^N m_i(u_i - v_i) \in [I_H]_{\leq d}$ and $\sum_{i=1}^N q_{N+1-i}(y_{N+1-i} - z_{N+1-i}) \in [I_H]_{\leq d}$. So if \mathcal{W}_N satisfies condition *i*), the binomial $w_{N_b} - w_{N_r} \in [I_H]_{\leq d}$, and thus, $p_w \in [I_H]_{\leq d}$.

To finish the proof, assume that S_N and R_N share an edge, e . Then the claim above becomes:

$$p_w = \sum_{i=1}^N m_i(u_i - v_i) + t_e \left(\frac{m_N v_N}{t_e} - \frac{q_N y_N}{t_e} \right) + \sum_{i=1}^N q_{N+1-i}(y_{N+1-i} - z_{N+1-i})$$

and we just need to show that, in fact, t_e divides $m_N v_N$ and $q_N y_N$. But this is clear to see since $e \in S_N$ which implies $t_e | v_N$ and $e \in R_N$ which implies $t_e | y_N$.

□

Example 5.2 (Independence models). Let H be the complete k -partite hypergraph with d vertices in each partition V_1, \dots, V_k . These hypergraphs correspond to the independence model in statistics. Equivalently, the edge subring of the complete k -partite hypergraph with d vertices in each partition parametrizes the Segre embedding of $\mathbb{P}^d \times \dots \times \mathbb{P}^d$ with k copies.

The ideal I_H is generated by quadrics. To see this, let \mathcal{W} , $\text{supp}(\mathcal{W}) \subseteq H$, be a primitive monomial walk of length $2n$, $n > 2$. Choose a multiset $E' \subset \mathcal{W}$ consisting of $n-1$ blue and $n-1$ red edges. Since each edge must contain a vertex from each V_i , for each i , there is at most one vertex in $V(E') \cap V_i$ that is not covered by a red edge and a blue edge from E' . Consequently, $V(E')$ contains a vertex from each V_i that belong to at least one red edge and at least one blue edge of E' .

For a multiset of edges, M , with $\text{supp}(M) \subseteq H$, we define the *max degree* of a vertex:

$$\text{maxdeg}(v; M) := \max(\deg_{red}(v; M), \deg_{blue}(v; M)).$$

The partitioning of the vertices ensures that $V(E')$ cannot contain more than k vertices whose *maxdeg* with respect to E' is $n-1$. Indeed, if there are more than k vertices with *maxdeg* equal to $n-1$, then two of those vertices must belong to the same partition, V_j . This would imply that \mathcal{W} contains at least $4(n-1)$ edges, which is impossible when $n > 2$.

Next, choose $n-1$ new blue edges and $n-1$ red edges in the following manner:

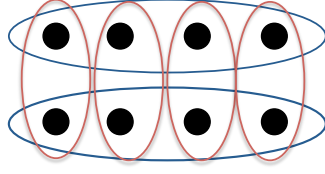


FIGURE 4.

Let $d_b(v) := \deg_{blue}(v; E')$ and $d_r(v) := \deg_{red}(v; E')$. For $i = 1, \dots, k$ choose a vertex from $V(E'_{blue}) \cap V(E'_{red}) \cap V_i$ that has the largest maxdeg with respect to E' ; let b_{n-1} and r_{n-1} be this set of vertices. For all $v \in b_{n-1}$, reduce $d_b(v)$ and $d_r(v)$ by 1. Now choose b_1, \dots, b_{n-2} by the following algorithm:

```

for i from 1 to k do:
    let  $V_i := \text{sort } V(E') \cap V_i$  by  $d_b(v)$  in decreasing order;
for j from  $n-2$  down to 1 do:
    (
         $b_j := \text{list } \{v_i : v_i \text{ is first element in } V_i\}$ ;
        for all  $v \in b_j$  do  $d_b(v) = d_b(v) - 1$ ;
        for i from 1 to k do  $V_i = \text{sort } V_i$  by  $d_b(v)$  in decreasing order;
    ).

```

Let $R_1 = \{b_1, \dots, b_{n-1}\}$ and $S_1 = \{r_1, \dots, r_{n-1}\}$. Then R_1 and S_1 are red and blue splitting sets of \mathcal{W} that share an edge. Thus, condition ii) of Theorem 5.1 is met, and consequently I_H is generated in degree 2.

When H is a non-uniform hypergraph, the toric ideal I_H is not necessarily homogeneous. For example, Figure 4 supports a binomial in I_H where H consists of edges of size two and four; note that the edges still satisfy the balancing condition (*). However, we can still modify the conditions of Theorem 5.1 to find degree bounds for the toric ideals of non-uniform hypergraphs. Proposition 5.3 gives a prescription for determining a degree bound on the generators of I_H in terms of local structures of H .

Proposition 5.3. *Given a hypergraph H and a binomial $f_{\mathcal{E}} \in I_H$ arising from the balanced edge set \mathcal{E} with $n = |\mathcal{E}_{blue}| \geq |\mathcal{E}_{red}|$, $f_{\mathcal{E}}$ is a linear combination of binomials in I_H of degree less than n if one of the following two conditions hold:*

i) *there exists a proper splitting set S of \mathcal{E} with decomposition (Γ_1, S, Γ_2) where $|\Gamma_{i_{blue}}|, |\Gamma_{i_{red}}| < n$ for $i = 1, 2$,*

or

ii) *there is a pair of blue and red splitting sets of \mathcal{E} , S and R , of size less than n with decompositions $(\Gamma_1, S, \Gamma_2), (\Upsilon_1, R, \Upsilon_2)$ such that $|\Gamma_{1_{blue}}|, |\Upsilon_{2_{red}}| < n$, $|\Gamma_{2_{blue}}|, |\Upsilon_{1_{red}}| \leq n$, and $S \cap R \neq \emptyset$.*

Proof. This proof follows the proof of sufficiency for Theorem 5.1. Note that in the proof, the uniform condition doesn't play an essential role; it is only invoked to bound the size of the red and blue parts of each monomial hypergraph appearing in the decompositions

involved. Thus, the hypothesis of Proposition 5.3 acts in place of the uniform condition in Theorem 5.1. \square

We close with an application.

For the remainder of this section, we will concern ourselves with the first tangential variety, $\text{Tan}((\mathbb{P}^1)^n)$. In [18], Sturmfels and Zwiernik use cumulants to give a monomial parameterization of $\text{Tan}((\mathbb{P}^1)^n)$. The variety $\text{Tan}((\mathbb{P}^1)^n)$ is associated to a class of hidden subset models [18, Example 5.2], and context-specific independence models [11]. We now derive a bound for the toric ideal of the image of $\text{Tan}((\mathbb{P}^1)^n)$ in higher cumulants and, equivalently, for the Markov complexity of these models.

Example 5.4. Let $H = (V, E)$ where $V = \{1, \dots, n\}$ and $E = \{e : e \subseteq V \text{ and } |e| \geq 2\}$. Then the set of polynomials vanishing on the image of $\text{Tan}((\mathbb{P}^1)^n)$ in higher cumulants is the toric ideal I_H (see [18, Theorem 4.1]).

The hypergraph in Example 5.4 is the complete hypergraph on n vertices after removing all singleton edges. The degree bound on the generators of this hypergraph can be found by looking at a smaller hypergraph.

Lemma 5.5. *Let $H_1 = (V, E_1)$ where $V = \{1, \dots, n\}$ and $E_1 = \{e : e \subseteq V \text{ and } |e| \geq 2\}$, and let $H_2 = (V, E_2)$ where $E_2 = \{e \subseteq V : 2 \leq |e| \leq 3\}$. If the ideal I_{H_2} is generated in degree at most d , then the ideal I_{H_1} is generated in degree at most d .*

Proof. Consider I_{H_2} as an ideal in the bigger polynomial ring $S := k[t_{e_i} : e_i \in H_1]$, denoted as $\tilde{I}_{H_2} := I_{H_2}S$. Assume that I_{H_2} , and consequently, \tilde{I}_{H_2} , is generated in degree at most d . Pick an arbitrary binomial

$$u - v = t_{e_{i_1}} t_{e_{i_2}} \cdots t_{e_{i_n}} - t_{e_{j_1}} t_{e_{j_2}} \cdots t_{e_{j_m}} \in I_{H_1}.$$

Since every edge $e \in H_1$ is the disjoint union of a collection of edges $e_{k_1}, \dots, e_{k_l} \in H_2$, we may write $t_e - \prod_{i=1}^l t_{e_{k_i}} \in I_{H_1}$. Noting that

$$t_e - \prod_{i=1}^l t_{e_{k_i}} = (t_e - t_{e_{k_1}} t_{\cup_{i=2}^l e_{k_i}}) - \sum_{j=1}^{l-2} \left[\left(\prod_{i=1}^j t_{e_{k_i}} \right) (t_{\cup_{i=j+1}^l e_{k_i}} - t_{e_{j+1}} t_{\cup_{i=j+2}^l e_{k_i}}) \right],$$

one easily sees that the binomial $t_e - \prod_{i=1}^l t_{e_{k_i}}$ is generated by quadratics. In turn, this essentially shows that relations in I_{H_2} allow us to rewrite $u - v$ in terms of edges $e_{i_1}, \dots, e_{i_n}, e_{j_1}, \dots, e_{j_m} \in E_2$ of size 2 and 3 only. The claim follows since $u - v$ can be expressed as a binomial in \tilde{I}_{H_2} . \square

Theorem 5.6. *Let $H = (V, E)$ where $V = \{1, \dots, n\}$ and $E = \{e \subseteq V : 2 \leq |e| \leq 3\}$. The toric ideal of H is generated by quadratics and cubics.*

In particular the image of $\text{Tan}((\mathbb{P}^1)^n)$ in higher cumulants is generated in degrees 2 and 3.

In the following proof, we examine the local combinatorics of H to illustrate how the structure of a hypergraph reveals insights into the generating set of I_H .

Proof. Let $f_{\mathcal{E}}$ be a primitive binomial in I_H with \mathcal{E} a balanced edge set. Without loss of generality, we will assume throughout the proof $|\mathcal{E}_{\text{blue}}| \geq |\mathcal{E}_{\text{red}}|$. If \mathcal{E} contains only 2-edges or only 3-edges, then by [16, Theorem 14.1] $f_{\mathcal{E}}$ is a linear combination of quadratics. So we will assume \mathcal{E} contains a 2-edge and a 3-edge.

Since $|\mathcal{E}_{blue}| \geq |\mathcal{E}_{red}|$, \mathcal{E}_{blue} must contain at least as many 2-edges as \mathcal{E}_{red} , and in order to satisfy (*), the difference between the number of 3-edges in \mathcal{E}_{red} and the number of 3-edges in \mathcal{E}_{blue} must be a multiple of 2.

Notice that for every pair e_1, e_2 of 3-edges (where e_1 and e_2 do not need to be unique), there are three 2-edges in H , e_3, e_4, e_5 , such that

$$\{e_1, e_2\} \sqcup_m \{e_3, e_4, e_5\}$$

is a balanced edge set. Let $\mathcal{B}_{2,3} \subset I_H$ be the set of all binomials arising from balanced edge sets of this form. Then $f_{\mathcal{E}}$ is a linear combination of binomials in $\mathcal{B}_{2,3}$ and $f_{\mathcal{E}'}$, where \mathcal{E}'_{blue} and \mathcal{E}'_{red} contains the same number of 2-edges and exactly one 3-edge.

Since it suffices to consider primitive binomials, we will proceed inductively by showing that every primitive degree n binomial in

$$\mathcal{B}_h := \{f_{\mathcal{E}} \in I_H : |\mathcal{E}_{blue}| = |\mathcal{E}_{red}| \text{ and } \mathcal{E}_{blue}, \mathcal{E}_{red} \text{ contain exactly one 3-edge each}\}$$

is a linear combination of binomials in \mathcal{B}_h with degree less than n .

Let $f_{\mathcal{E}} \in \mathcal{B}_h$ such that $\text{degree } f_{\mathcal{E}} = n > 3$ and $f_{\mathcal{E}}$ is primitive. Let e_1 be the 3-edge in \mathcal{E}_{red} . Since $f_{\mathcal{E}}$ is primitive, e_1 must intersect a 2-edge e_2 in \mathcal{E}_{blue} . Let $e_2 = \{v_1, v_2\}$ where $v_1 \in e_1$.

The edge e_2 intersects at most one other edge of \mathcal{E}_{red} besides e_1 . We will examine the possible intersections of e_2 and \mathcal{E}_{red} in order to find splitting sets of \mathcal{E} that satisfy one of the conditions listed in Proposition 5.3. For illustrations of Case 1 and Case 3 see Figures 5 and 6. In all three cases, we will construct S , Γ_1 and Γ_2 such that S is a splitting set of \mathcal{E} with an associated decomposition (Γ_1, S, Γ_2) which satisfies the properties of condition *i*) in Theorem 5.3. In fact, $f_{\mathcal{E}}$ will be a linear combination of f_{Γ_1} and f_{Γ_2} , both of which have strictly lower degree than $f_{\mathcal{E}}$. Furthermore, since the blue and red parts of Γ_1 and Γ_2 will contain the same number of 2 and 3-edges, it follows that $f_{\Gamma_1}, f_{\Gamma_2} \in \mathcal{B}_h$.

Case 1: The edge $e_1 = e_2 \cup \{v_3\} = \{v_1, v_2, v_3\}$ for some $v_3 \in V(\mathcal{E})$.

Since $v_3 \notin e_2$ and $|\mathcal{E}_{blue}| = |\mathcal{E}_{red}|$, there must be a 2-edge $e_3 \in \mathcal{E}_{red}$ such that $v_3 \notin e_3$ in order for (*) to hold. Let $e_3 = \{v_4, v_5\}$ and $e_4 = \{v_3, v_4, v_5\}$. The sets S , Γ_1 and Γ_2 in this case are:

$$\begin{aligned} S &= \{e_4\} \\ \Gamma_1 &= (\mathcal{E}_{blue} - \{e_2\}) \sqcup_m ((\mathcal{E}_{red} - \{e_1, e_3\}) \sqcup \{e_4\}) \\ \Gamma_2 &= \{e_2, e_4\} \sqcup_m \{e_1, e_3\}. \end{aligned}$$

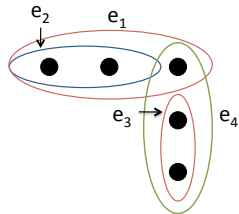


FIGURE 5. Case 1

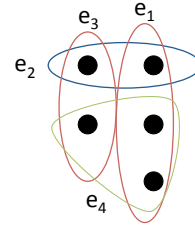


FIGURE 6. Case 3

Case 2: The edge $e_1 = \{v_1, v_3, v_4\}$ for some $v_3, v_4 \in V(\mathcal{E})$ and there is a 2-edge $e_3 \in \mathcal{E}_{red}$ such that $e_3 = \{v_2, v_3\}$.

Since $v_3 \notin e_2$, $\deg_{\text{blue}}(v_3; \mathcal{E}) = \deg_{\text{red}}(v_3; \mathcal{E}) \leq n - 1$ and, thus, there exists a 2-edge $e_4 \in \mathcal{E}_{\text{red}}$ such that $v_3 \notin e_4$. Let $e_4 = \{v_5, v_6\}$.

Now let $e_5 = \{v_3, v_4, v_5\}$ and $e_6 = \{v_3, v_6\}$. The sets S , Γ_1 and Γ_2 in this case are:

$$\begin{aligned} S &= \{e_5, e_6\} \\ \Gamma_1 &= (\mathcal{E}_{\text{blue}} - \{e_2\}) \sqcup_m ((\mathcal{E}_{\text{red}} - \{e_1, e_3, e_4\}) \sqcup \{e_5, e_6\}) \\ \Gamma_2 &= \{e_2, e_5, e_6\} \sqcup_m \{e_1, e_3, e_4\}. \end{aligned}$$

Case 3: There is a 2-edge $e_3 \in \mathcal{E}_{\text{red}}$ such that $v_2 \in e_3$ and $e_2 \cap e_3 = \emptyset$. In this case, let $e_4 = (e_1 - \{v_1\}) \cup (e_3 - \{v_2\})$. The sets S , Γ_1 and Γ_2 in this case are:

$$\begin{aligned} S &= \{e_4\} \\ \Gamma_1 &= (\mathcal{E}_{\text{blue}} - \{e_2\}) \sqcup_m ((\mathcal{E}_{\text{red}} - \{e_1, e_3\}) \sqcup \{e_4\}) \\ \Gamma_2 &= \{e_2, e_4\} \sqcup_m \{e_1, e_3\}. \end{aligned}$$

□

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